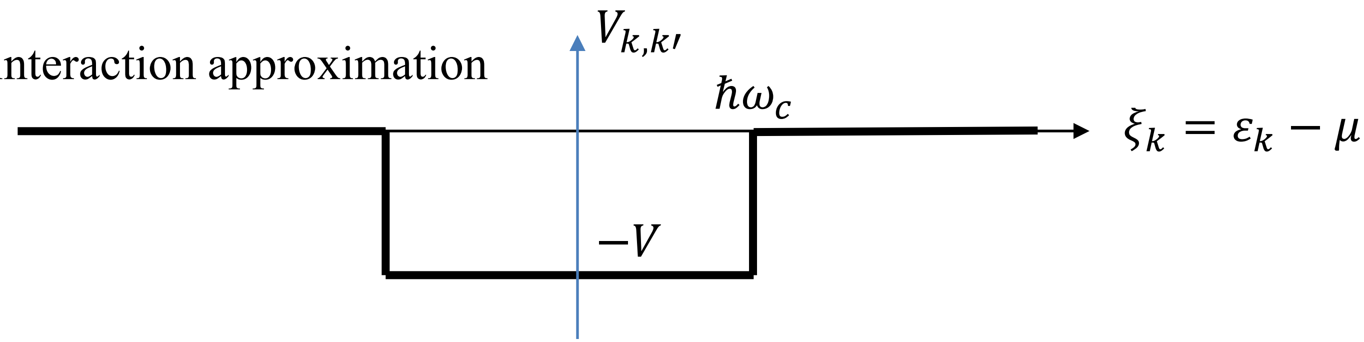
Eliashberg Theory of Strong-Coupled Superconductors

Cooper pairing interaction approximation



What are $\hbar\omega_c$ and V?

Can we go beyond the approximation that $\mathrm{D}(E_F)V \ll 1$?

In the strong electron-phonon coupling limit, the single particle states (k,σ) are no longer good eigenstates. These states are lifetime broadened by phonon emission.

Treat the gap Δ as a complex function of energy. The energy-dependent phase is distinct from that of the coherent BCS gap.
$Im[\Delta(E)]$ is due to the decay of quasiparticles and the creation of real phonons
$\mathrm{Re}[\Delta(E)]$ goes through resonant absorption when $E \sim \hbar\omega_{phonon}$

Eliashberg Theory of Strong-Coupled Superconductors

Self-consistent gap equation

$$\Delta(i\omega_n)Z(i\omega_n)=\pi T\sum_m [\lambda(i\omega_m-i\omega_n) -\mu^*(\omega_c)\theta(\omega_c-|\omega_m|)] \times\frac{\Delta(i\omega_m)}{\sqrt{\omega_m^2+\Delta^2(i\omega_m)}}$$

Renormalization factors

$$Z(i\omega_n)=1+\frac{\pi T}{\omega_n}\sum_m \lambda(i\omega_m-i\omega_n)\frac{\omega_m}{\sqrt{\omega_m^2+\Delta^2(i\omega_m)}}$$

Pairing strength

$$\lambda(i\omega_m-i\omega_n)=2\int_0^\infty \frac{\Omega\alpha^2F(\Omega)d\Omega}{\Omega^2+(\omega_n-\omega_m)^2}\equiv\lambda(m-n)$$

Reduces to BCS with …

$$\lambda(i\omega_m-i\omega_n)=\begin{cases}\lambda & \text{for both } |\omega_n|,|\omega_m|<\omega_c\\ 0 & \text{otherwise ,}\end{cases}$$

(λ is like $-V$)

$$\lambda=\int_0^\infty \frac{2\alpha^2F(\Omega)d\Omega}{\Omega}$$

α^2 is the electron-phonon coupling
$F(\Omega)$ is the phonon spectrum

$$Z(i\omega_n)=1+\lambda$$

$$\Delta(i\omega_n)=\begin{cases}\Delta(T), & |\omega_n|<\omega_c\ ,\\ 0, & |\omega_n|>\omega_c\ ,\end{cases}$$

with

$$\Delta(T)=\frac{\lambda-\mu^*}{1+\lambda}\pi T\sum_{\substack{m\\|\omega_m|<\omega_c}}\frac{\Delta(T)}{\sqrt{\omega_m^2+\Delta^2(T)}}$$

$$N(0)V\equiv\frac{\lambda-\mu^*}{1+\lambda}$$

Matsubara frequencies: $i\omega_n = i\pi k_B T(2n-1)$ with $n = 0, \pm1, \pm2, \ldots$

J. P. Carbotte, Rev Mod Phys **62, 1027 (1990)**

Strong-Coupled Superconductors

With strong electron-phonon coupling, the Cooper pairs and quasiparticles have a finite lifetime. This is modeled by introducing a "gap function" $\Delta(\omega)$ which is both complex and frequency dependent.

T_c is enhanced by strong-coupling effects:

$$T_c = \frac{\hbar\omega_{\ell n}}{1.2k_B}\exp\left(\frac{-1.04(1+\lambda)}{\lambda-\mu^*(1+0.62\lambda)}\right)$$

As opposed to BCS weak coupling:

$$T_c \cong \hbar\omega_D e^{-1/(\lambda-\mu*)}$$

$$D(0)V = \lambda - \mu^*$$

where $\omega_{\ell n}$ is used as an average phonon frequency, and it and λ are defined by

$$\omega_{\ln} \equiv \exp\left[\frac{2}{\lambda}\int_0^\infty d\nu \ln(\nu)\frac{\alpha^2(\nu)F(\nu)}{\nu}\right] \approx e^{\langle ln\omega\rangle}$$

Electron-Phonon coupling

Phonon DOS

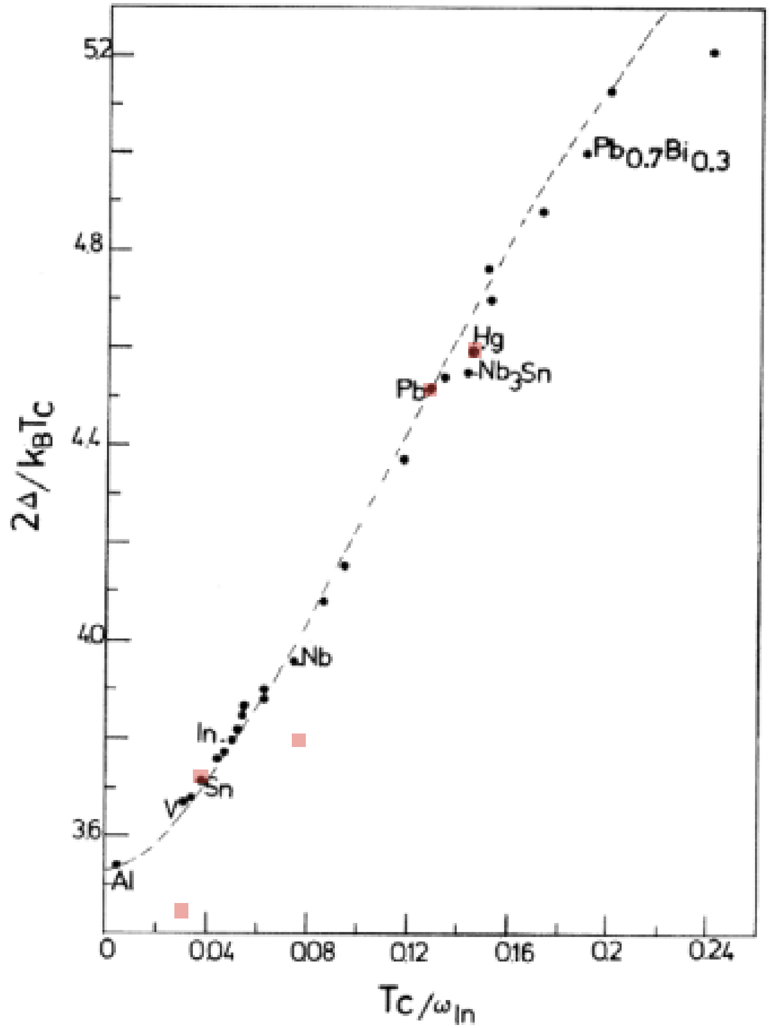
$$\lambda \equiv 2\int_0^\infty d\nu \frac{\alpha^2(\nu)F(\nu)}{\nu}$$ is called the McMillan parameter.

$\alpha^2(\omega)\mathrm{F}(\omega)$ is called the Eliashberg function.

$$\mu^* = \frac{\mu}{1+\mu\ln(\frac{\epsilon_F}{\hbar\omega_D})}$$ (more about Coulomb repulsion below)

Strong-Coupling Correction to Gap Ratio

$$\frac{2\Delta_0}{(k_B T_c)} = 3.53\left[1 + 12.5\left(\frac{T_c}{\omega_{\ln}}\right)^2 \ln\left(\frac{\omega_{\ln}}{2T_c}\right)\right]$$



$$\omega_{\ln} \equiv \exp\left[\frac{2}{\lambda}\int_0^\infty d\nu \, \ln(\nu)\frac{\alpha^2(\nu)F(\nu)}{\nu}\right]$$

$$\approx e^{\langle ln\omega\rangle}$$

Fig. 4. The gap ratio $2\Delta_0/(k_B T_c)$ as a function of $T_c/\omega_{\ell n}$. The black circles indicate theoretical calculations, with some of the elements and a couple of binary alloys indicated. The unmarked circles refer mostly to various binary alloys [57]. These calculations use an electron–phonon spectral function $\alpha(\nu)^2F(\nu)$ and value of μ^* extracted from tunneling experiments, or, in some cases taken from calculations [58,59]. Selected experimental values are indicated with red squares. Note the excellent agreement of theory with experiment in the case of Sn, Pb and Hg, with more deviation in the case of vanadium and niobioum. Sources are available in Ref.

The Eliashberg Function

Electron-phonon scattering from *k* to *k'* with creation of a phonon $\hbar\omega_{\lambda,k'-k}$ with polarization λ

$$\alpha^2(\Omega)F(\Omega) = \frac{\int \frac{dS_{k'}}{|\vec{v}_{k'}|} \int \frac{dS_k}{|\vec{v}_k|} \frac{1}{(2\pi)^3\hbar} \sum_\lambda |g_{k',k,\lambda}|^2 \delta[\Omega - \omega_{\lambda,k'-k}]}{\int \frac{dS_k}{|\vec{v}_k|}}$$

Element of Fermi surface area

Group velocity on the Fermi surface

$$\lambda \equiv 2\int_0^\infty d\nu \frac{\alpha^2(\nu)F(\nu)}{\nu}$$

is a dimensionless measure of the strength of electron-phonon coupling.
Ranges from 0.1 to 1.7 in various metals

DOS at E_F

Fermi surface average of el-ph matrix element²

$$\lambda \cong \frac{N(0)\langle I^2\rangle}{M\langle\omega^2\rangle}$$

Ionic mass

Mean-square phonon frequency

Weak-coupling BCS Approx:
$\lambda << 1$

Predictions for λ in the Strong Coupling Limit

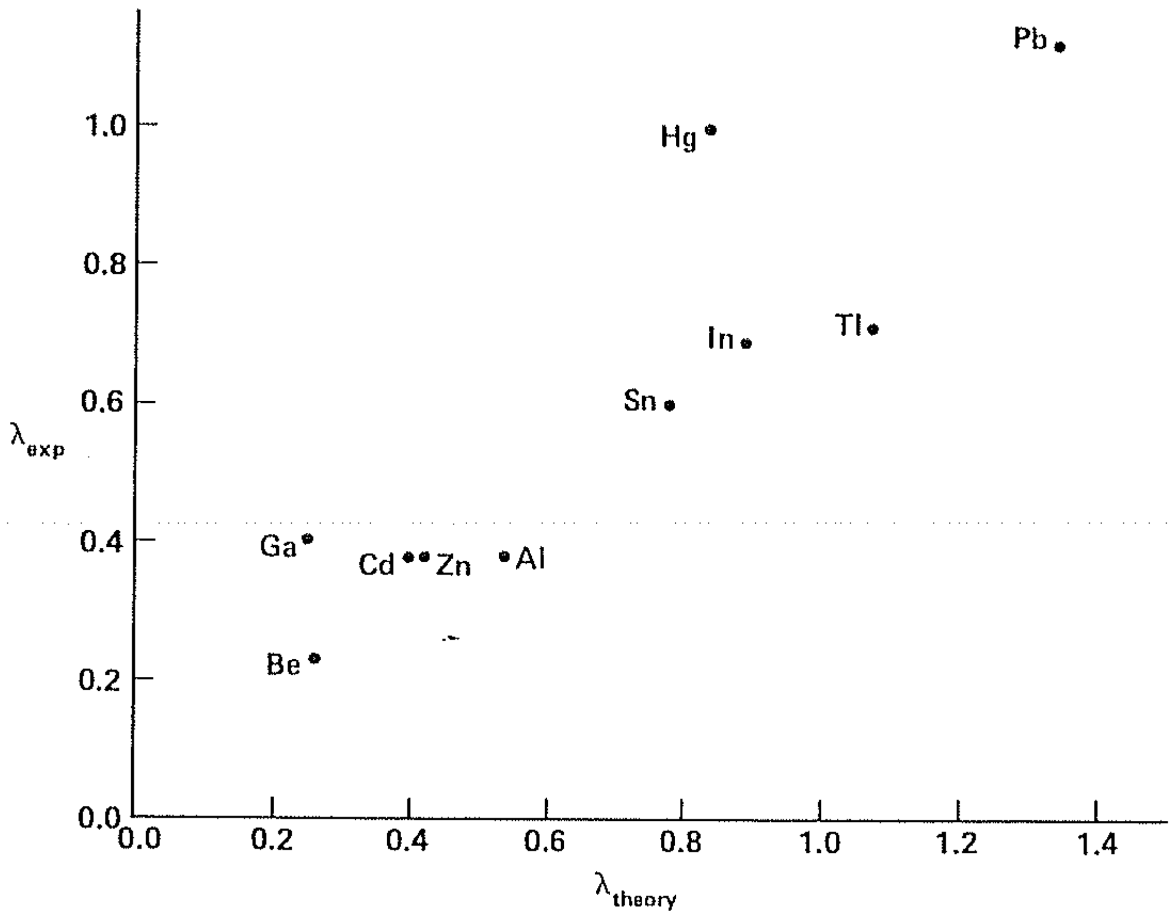


FIG. III.8. Comparison of the theoretical electron–phonon coupling constants obtained from pseudopotentials with those obtained empirically using McMillan's formula.

Predictions for T_c in the Strong Coupling Limit

In the strong-coupling limit:

$$T_c \sim \sqrt{\lambda\langle\omega^2\rangle} \sim \sqrt{\frac{1}{M}}$$

where *M* is the ionic mass. This argues for materials light masses (hydrogen)

Allen and Dynes, Phys. Rev. B 12, 905 (1975)

$$T_c = 0.183\sqrt{\lambda\langle\omega^2\rangle}$$ for $\lambda > 10$ and $\mu^* = 0$

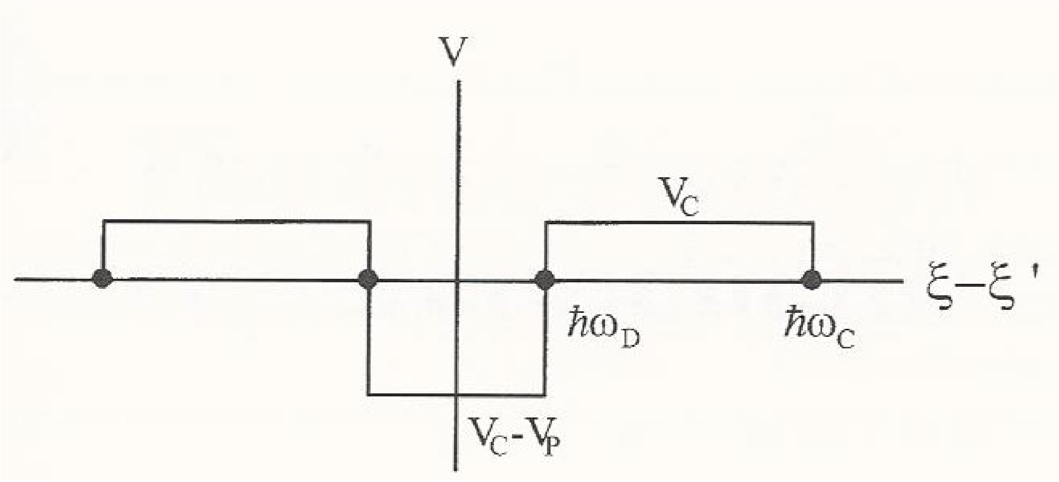
T_c increases with no saturation for very strong coupling!

	T_c (K)	$\langle\Omega\rangle$ (K)	$N(0)\langle I^2\rangle$	$\sqrt{\langle\Omega^2\rangle}$ (K)	λ
Nb	9.2	175	4.7	183	0.85
Nb_3Sn	18.1	146	7.9	163	1.67
Pb	7.2	60	2.4	65	1.55

Prediction for Isotope Exponent α in the Strong Coupling Limit

$$T_c M^{\alpha} = constant$$

$$\alpha = \frac{1}{2}\left[1 - \left(\mu^* \ln \frac{\langle \Omega \rangle}{1.20 T_c}\right)^2 \frac{1 + 0.62\lambda}{1 + \lambda}\right]$$



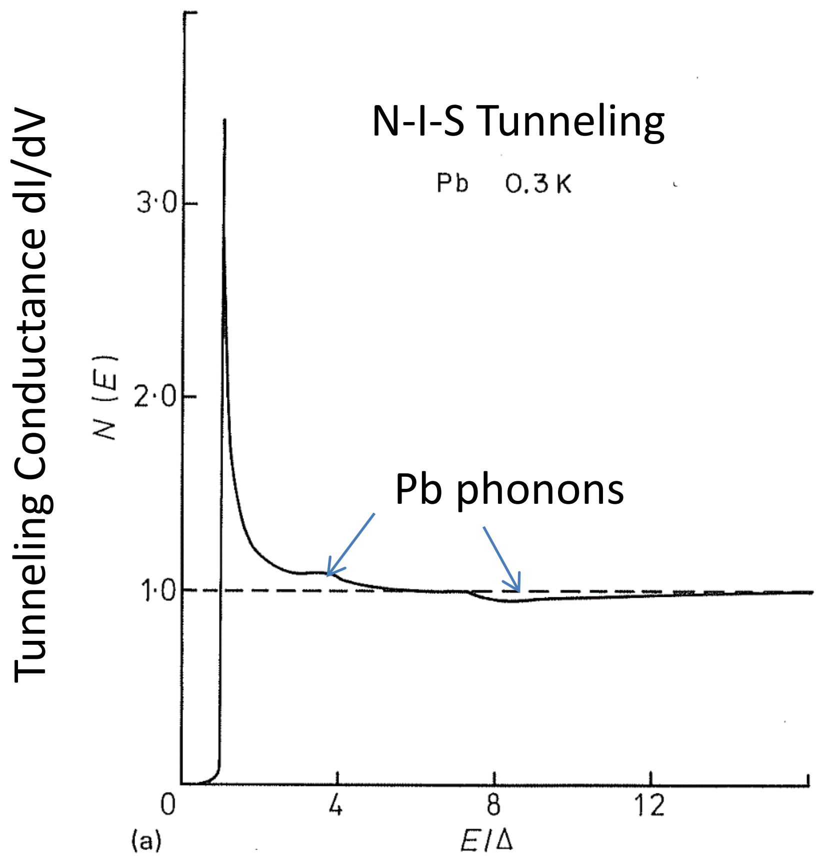
$$\mu^* = \frac{\mu}{1 + \mu \ln(\frac{\epsilon_F}{\hbar\omega_D})}$$

$$\lambda_{BCS,weak} = D(0)V_p$$

$$\mu = D(0)V_C$$

$$\mu^* = \frac{\mu}{1 + \mu \, ln(\omega_C/\omega_D)}$$

Tunneling Spectroscopy and the Eliashberg Function



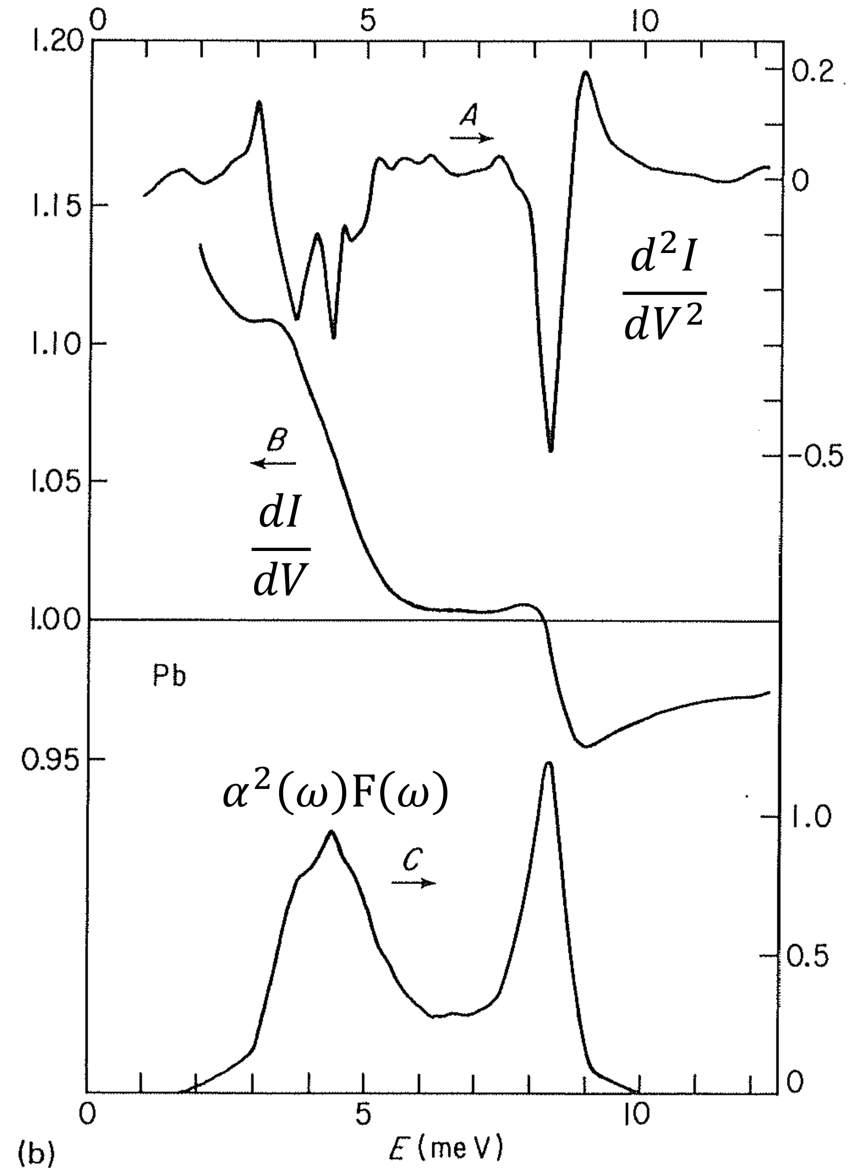
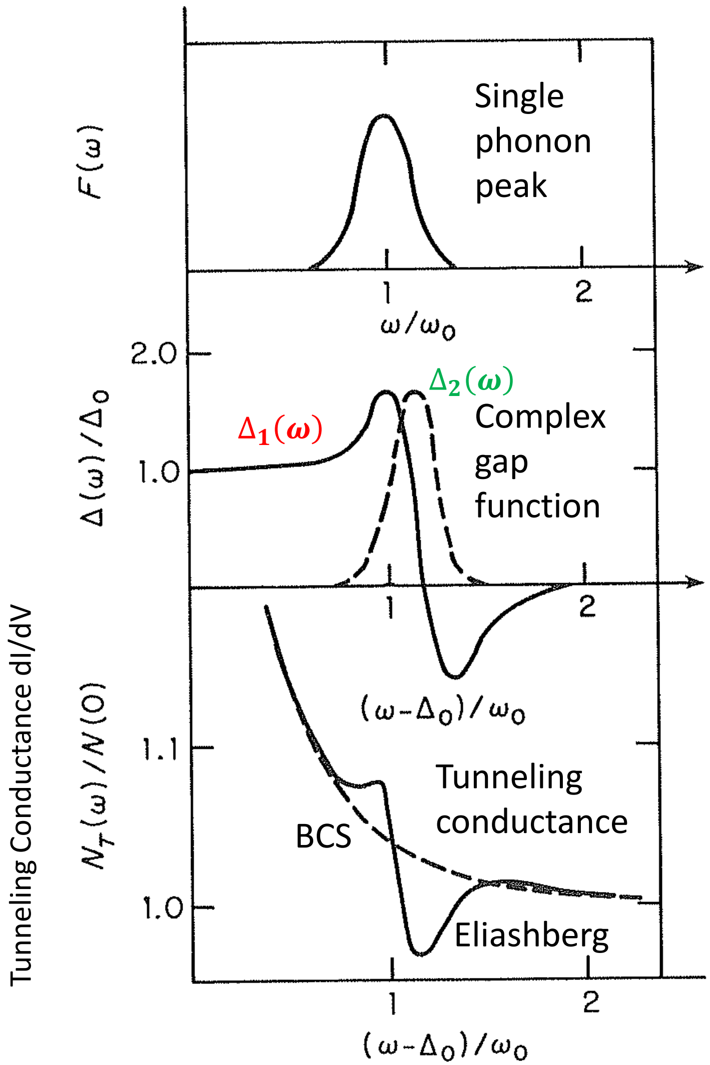


Fig. 1.6. (a) Normalized conductance of a tunnel junction involving lead at 0.3 K (after Giaever, Hart, and Megerle, 1962). Note the extremely sharp energy gap. The small deviations of the density of states from unity in the 4–10 mV range are due to the phonons of lead. (b) Illustration of the use of tunneling to determine the effective phonon spectrum $\alpha^2F(\omega)$ of a strong-coupling superconductor. The Pb phonons are revealed in detail by the analysis of McMillan and Rowell (1965). Curves *A*, *B*, and *C*, respectively, show the second derivative, first derivative, and effective phonon spectrum for lead.

Extracting the Eliashberg Function from Tunneling Spectroscopy Data



$$\Delta(\omega) = \Delta_1(\omega) + i\,\Delta_2(\omega)$$

$\Delta_2(\omega)$ ~ 1 / lifetime of excitations

$\Delta_2(\omega)$ is large when phonon emission is possible

DOS with complex Δ

$$N(\omega) = \mathrm{Re}\left\{\frac{|\omega|}{[\omega^2 - \Delta^2(\omega)]^{1/2}}\right\}$$

Tunneling Spectroscopy and the Eliashberg Function

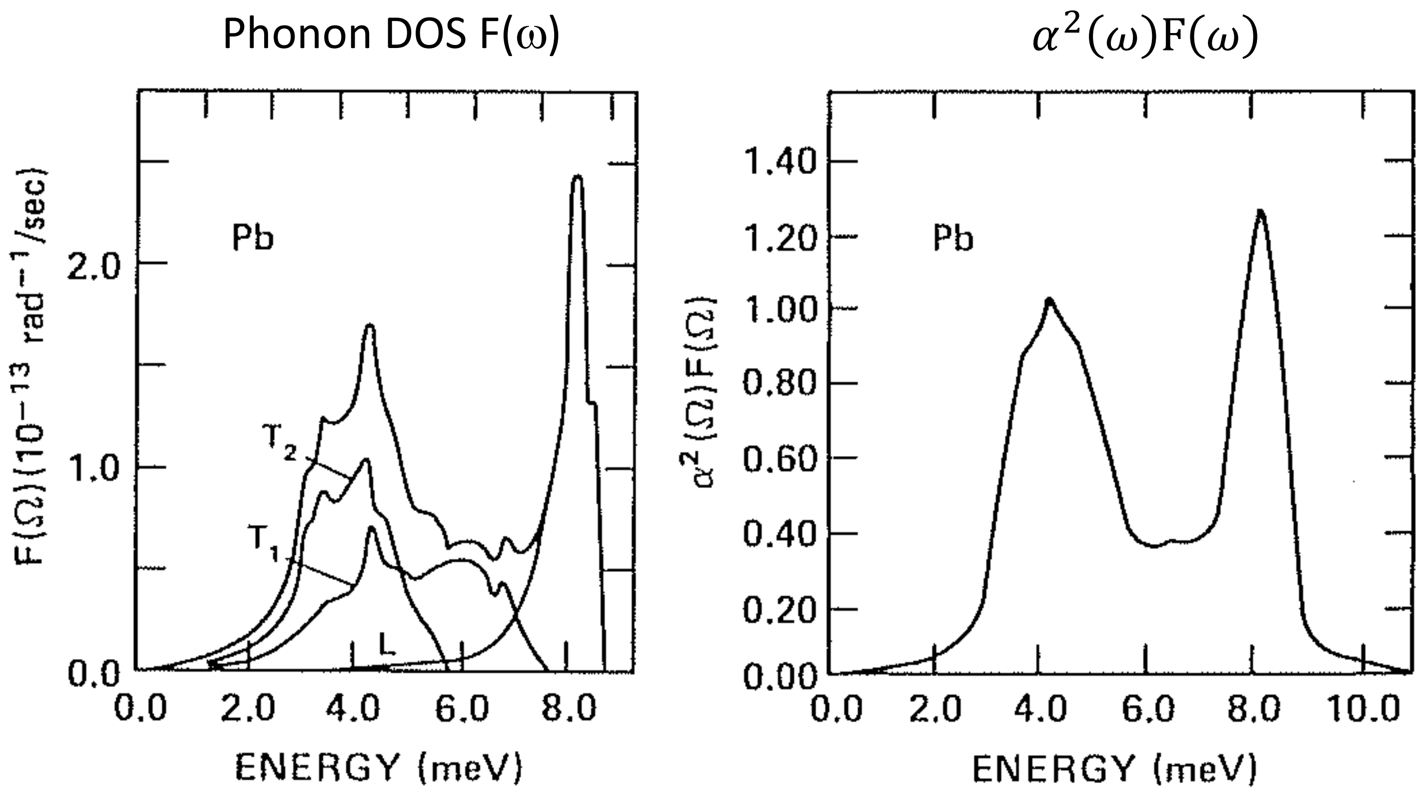
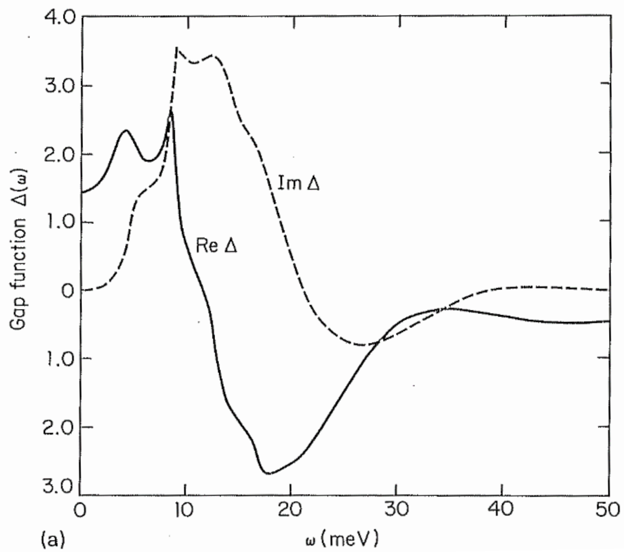


FIG. III.6. Comparison of the phonon density of states of Pb as obtained from (a) neutron scattering (after Stedman *et al.*[18]) with that obtained from (b) electron tunneling (after McMillan and Rowell[17]).

Extracting the Eliashberg Function from Tunneling Spectroscopy Data

Fig. 4.5. (a) The real and imaginary parts of the computed gap function $\Delta(\omega)$ for lead obtained from the data of McMillan and Rowell (1969). In this figure, the dashed curve is the imaginary part and the solid curve is the real part of the gap function.



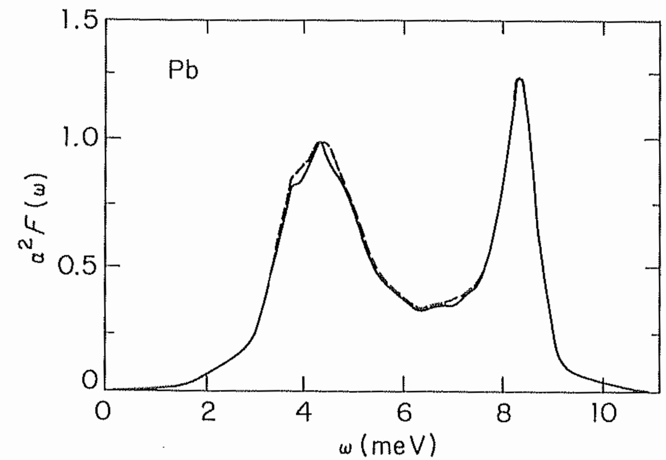


Fig. 4.4. A comparison of the $\alpha^2F(\omega)$ functions for lead obtained from the data of McMillan and Rowell (1969) as reduced using the variational scheme (dashed curve) and using the nonvariational scheme of Galkin, D'yachenko, and Svistunov (1974) (solid curve). (After Galkin, D'yachenko, and Svistunov, 1974.)

References

E. L. Wolf, *Principles of Electron Tunneling Spectroscopy*, (Oxford University Press, New York, 1989) page 159 ff

Webb, G. W. and Marsiglio, F. and Hirsch, J. E., Superconductivity in the elements, alloys and simple compounds, Physica C 514, 17-27 (2015).

M. L. Cohen in *BCS: 50 Years*, (World Scientific, 2011), page 375

Robert M. White and Theodore H. Geballe, *Long Range Order in Solids*, (Academic Press, 1979), page 110

J. B. Ketterson and S. N. Song, *Superconductivity*, (Cambridge Univ. Press, 1999), page 240

J. P. Carbotte, "Properties of boson-exchange superconductors," Rev Mod Phys **62, 1027 (1990)**